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# UNIQUE DECOMPOSITION FOR A POLYNOMIAL OF LOW RANK

EDOARDO BALLICO, ALESSANDRA BERNARDI

## ABSTRACT

Let  $F$  be a homogeneous polynomial of degree  $d$  in  $m+1$  variables defined over an algebraically closed field of characteristic 0 and suppose that  $F$  belongs to the  $s$ -th secant variety of the  $d$ -uple Veronese embedding of  $\mathbb{P}^m$  into  $\mathbb{P}^{\binom{m+d}{d}-1}$  but that its minimal decomposition as a sum of  $d$ -th powers of linear forms requires more than  $s$  addenda. We show that if  $s \leq d$  then  $F$  can be uniquely written as  $F = M_1^d + \cdots + M_t^d + Q$ , where  $M_1, \dots, M_t$  are linear forms with  $t \leq (d-1)/2$ , and  $Q$  a binary form such that  $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$  with  $l_i$ 's linear forms and  $m_i$ 's forms of degree  $d_i$  such that  $\sum (d_i + 1) = s - t$ .

## INTRODUCTION

In this paper we will always work with an algebraically closed field  $K$  of characteristic 0. Let  $X_{m,d} \subset \mathbb{P}^N$ , with  $m \geq 1$ ,  $d \geq 2$  and  $N := \binom{m+d}{d} - 1$ , be the classical Veronese variety obtained as the image of the  $d$ -uple Veronese embedding  $\nu_d : \mathbb{P}^m \rightarrow \mathbb{P}^N$ . The  $s$ -th secant variety  $\sigma_s(X_{m,d})$  of the Veronese variety  $X_{m,d}$  is the Zariski closure in  $\mathbb{P}^N$  of the union of all linear spans  $\langle P_1, \dots, P_s \rangle$  with  $P_1, \dots, P_s \in X_{m,d}$ . For any point  $P \in \mathbb{P}^N$ , we indicate with  $\text{sbr}(P) = s$  the minimum integer  $s$  such that  $P \in \sigma_s(X_{m,d})$ . This integer is called the *symmetric border rank* of  $P$ . Since  $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V^*)$ , with  $V$  an  $(m+1)$ -dimensional vector space over  $K$ , the generic element belonging to  $\sigma_s(X_{m,d})$  is the projective class of a form (a symmetric tensor) of type:

$$(1) \quad F = L_1^d + \cdots + L_r^d, \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}).$$

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem that is having a lot of attentions not only from classical Algebraic Geometry ([1], [7], [5], [6], [9]), but also from applications like Computational Complexity ([8]) and Signal Processing ([10]).

At the Workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented the following result:

([2], Corollary 1) Let  $F \in K[x_0, \dots, x_m]_d$  be such that  $\text{sbr}(F) + \text{sr}(F) \leq 2d+1$  and  $\text{sbr}(F) < \text{sr}(F)$ . Then there are an integer  $t \geq 0$ , linear forms  $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$ , and a form  $Q \in K[L_1, L_2]_d$  such that  $F = Q + M_1^d + \cdots + M_t^d$ ,  $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$ , and  $\text{sr}(F) = \text{sr}(Q) + t$ . Moreover  $t$ ,  $M_1, \dots, M_t$  and the linear span of  $L_1, L_2$  are uniquely determined by  $F$ .

In terms of tensors it can be translated as follows:

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([2], Corollary 2) Let  $T \in S^d V^*$  be such that  $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$  and  $\text{sbr}(T) < \text{sr}(T)$ . Then there are an integer  $t \geq 0$ , vectors  $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$ , and a symmetric tensor  $v \in S^d(\langle v_1, v_2 \rangle)$  such that  $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$ ,  $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$ , and  $\text{sr}(T) = \text{sr}(v) + t$ . Moreover  $t$ ,  $w_1, \dots, w_t$  and  $\langle v_1, v_2 \rangle$  are uniquely determined by  $T$ .

The natural questions that arose at that workshop from applied people, were about the possible uniqueness of the binary form  $Q$  in [2], Corollary 1 (ie. the vector  $v$  in [2], Corollary 2) and a bound on the number  $t$  of linear forms (ie. rank 1 symmetric tensors). We are finally able to give the most possible complete answer to this question. The main result of this paper is the following.

**Theorem 1.** *Let  $P \in \mathbb{P}^N$  with  $N = \binom{m+d}{d} - 1$ . Suppose that:*

$$\begin{aligned} \text{sbr}(P) &< \text{sr}(P) \text{ and} \\ \text{sbr}(P) + \text{sr}(P) &\leq 2d + 1. \end{aligned}$$

*Let  $\mathcal{S} \subset X_{m,d}$  be a 0-dimensional reduced subscheme that realizes the symmetric rank of  $P$ , and let  $\mathcal{Z} \subset X_{m,d}$  be a 0-dimensional non-reduced subscheme such that  $P \in \langle \mathcal{Z} \rangle$  and  $\deg \mathcal{Z} \leq \text{sbr}(P)$ . There is a unique rational normal curve  $C_d \subset X_{m,d}$  such that  $C_d \cap (\mathcal{S} \cup \mathcal{Z}) \geq d + 2$ . Then, for all points  $P \in \mathbb{P}^N$  as above we have that:*

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \quad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

*where  $\mathcal{S}_1 = \mathcal{S} \cap C_d$ ,  $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$  and  $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$ .*

*Moreover  $C_d$ ,  $\mathcal{S}_2$  and  $\mathcal{Z}$  are unique,  $\deg(\mathcal{Z}) = \text{sbr}(P)$ ,  $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ ,  $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$  and  $\mathcal{Z}$  is the unique zero-dimensional subscheme  $N$  of  $X_{m,d}$  such that  $\deg(N) \leq \text{sbr}(P)$  and  $P \in \langle N \rangle$ .*

In the language of polynomials, Theorem 1 can be rephrased as follows.

**Corollary 1.** *Let  $F \in K[x_0, \dots, x_m]_d$  be such that  $\text{sbr}(F) + \text{sr}(F) \leq 2d + 1$  and  $\text{sbr}(F) < \text{sr}(F)$ . Then there are an integer  $0 \leq t \leq (d-1)/2$ , linear forms  $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$ , and a form  $Q \in K[L_1, L_2]_d$  such that  $F = Q + M_1^d + \dots + M_t^d$ ,  $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$ , and  $\text{sr}(F) = \text{sr}(Q) + t$ .*

*Moreover the line  $\langle L_1, L_2 \rangle$ , the forms  $M_1, \dots, M_t$  and  $Q$  such that  $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$  with  $l_i$ 's linear forms and  $m_i$ 's forms of degree  $d_i$  such that  $\sum (d_i + 1) = s - t$ , are uniquely determined by  $F$ .*

An analogous corollary can be stated for symmetric tensors.

**Corollary 2.** *Let  $T \in S^d V^*$  be such that  $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$  and  $\text{sbr}(T) < \text{sr}(T)$ . Then there are an integer  $0 \leq t \leq (d-1)/2$ , vectors  $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$ , and a symmetric tensor  $v \in S^d(\langle v_1, v_2 \rangle)$  such that  $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$ ,  $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$ , and  $\text{sr}(T) = \text{sr}(v) + t$ . Moreover the line  $\langle v_1, v_2 \rangle$ , the vectors  $v_1, \dots, v_t$  and the tensor  $v$  such that  $v = \sum_{i=1}^q u_i^{\otimes (d-d_i)} \otimes z_i$  with  $u_i \in \langle v_1, v_2 \rangle$  and  $z_i \in S^{d_i}(\langle v_1, v_2 \rangle)$  such that  $\sum (d_i + 1) = s - t$ , are uniquely determined by  $T$ .*

Moreover in Theorem 2 and in Corollary 4, by introducing the notion of linearly general position of a scheme (Definition 1), we can also extend to a geometric description the condition for the uniqueness of the scheme  $\mathcal{Z}$  of Theorem 1. We can rephrase their contents in terms of homogeneous polynomials and symmetric tensors in the following Corollary.

**Corollary 3.** *Fix integers  $m \geq 2$  and  $d \geq 4$ . Fix  $F$  an homogeneous polynomial in  $m + 1$  variables of degree  $d$  ( $T \in S^d V$  respectively) such that  $\text{sbr}(F) \leq d$  ( $\text{sbr}(T) \leq d$ ). Let  $Z \subset \mathbb{P}^m$  be*

any smoothable zero-dimensional scheme such that  $\nu_d(Z)$  computes  $\text{sbr}(F)$  ( $\text{sbr}(T)$ ). Assume that  $Z$  is in linearly general position. Then  $Z$  is the unique scheme computing  $\text{sbr}(P)$  ( $\text{sbr}(F)$ ).

## 1. PROOFS

The existence of such a scheme  $\mathcal{Z}$  was known from [3] and [4] (see Remark 1 of [2]).

**Lemma 1.** Fix integers  $m \geq 2$  and  $d \geq 2$ , a line  $\ell \subset \mathbb{P}^m$  and any finite set  $E \subset \mathbb{P}^m \setminus \ell$  such that  $\sharp(E) \leq d$ . Then  $\dim(\langle \nu_d(E) \rangle) = \sharp(E) - 1$  and  $\langle \nu_d(\ell) \rangle \cap \langle \nu_d(E) \rangle = \emptyset$ .

*Proof.* Since  $h^0(\ell \cup E, \mathcal{O}_{\ell \cup E}(d)) = d + 1 + \sharp(E)$ , to get both statements it is sufficient to prove  $h^1(\mathcal{I}_{\ell \cup E}(d)) = 0$ . Let  $H \subset \mathbb{P}^m$  be a general hyperplane containing  $\ell$ . Since  $E$  is finite and  $H$  is general, we have  $H \cap E = \emptyset$ . Hence the residual exact sequence of the scheme  $\ell \cup E$  with respect to the hyperplane  $H$  is the following exact sequence on  $\mathbb{P}^m$ :

$$(2) \quad 0 \rightarrow \mathcal{I}_E(d-1) \rightarrow \mathcal{I}_{\ell \cup E}(d) \rightarrow \mathcal{I}_{\ell, H}(d) \rightarrow 0$$

Since  $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell, H}(d)) = 0$ , we get the lemma.  $\square$

*Proof of Theorem 1.* All the statements are contained in [2], Theorem 1, except the uniqueness of  $\mathcal{Z}$ , that  $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$  and that  $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ . Let  $\ell \subset \mathbb{P}^m$  be the line such that  $\nu_d(\ell) = C_d$ . Take  $Z, S, Z_i, S_i \subset \mathbb{P}^m$ ,  $i = 1, 2$ , such that  $\nu_d(Z) = \mathcal{Z}$ ,  $\nu_d(S) = \mathcal{S}$ ,  $\nu_d(Z_i) = \mathcal{Z}_i$ , and  $\nu_d(S_i) = \mathcal{S}_i$ . Assume the existence of another subscheme  $\mathcal{Z}' \subset X_{m,d}$  such that  $P \in \langle \nu_d(\mathcal{Z}') \rangle$  and  $\deg(\mathcal{Z}') \leq \text{sbr}(P)$ . Set  $\mathcal{Z}'_1 := \mathcal{Z}' \cap C_d$ . The proof of [2], Theorem 1, gives  $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup S_2$ . Since  $C_d$  is a smooth curve,  $\mathcal{Z}_1 \cup \mathcal{Z}'_1 \subset C_d$ ,  $S_2 \cap \ell = \emptyset$ , and  $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}'_1) \sqcup S_2$ , the schemes  $\mathcal{Z}$  and  $\mathcal{Z}'$  are curvilinear. Hence all subschemes of  $\mathcal{Z}$  and  $\mathcal{Z}'$  are smoothable. Hence any subscheme of either  $\mathcal{Z}$  or  $\mathcal{Z}'$  may be used to compute the border rank of some point of  $\mathbb{P}^N$ . Since  $\deg(\ell \cap (Z \cup S)) \geq d + 2$ ,  $\nu_d((Z \cup S) \cap \ell)$  spans  $\langle C_d \rangle$ . Lemma 1 implies  $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$ . Since  $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$  and  $\sharp(S) = \text{sr}(P)$ , we have  $P \notin \langle \mathcal{A} \rangle$  for any  $\mathcal{A} \subsetneq \mathcal{S}$ . Therefore we get that  $\langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle \mathcal{S}_1 \rangle$  is a unique point. Call  $P_1$  this point. Similarly,  $\langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_2 \rangle$  is a unique point and we call it  $P_2$ . Since  $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$ , the set  $\langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$  is at most one point. Since  $P_i \in \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ ,  $i = 1, 2$ , we have  $P_1 = P_2$  and  $\{P_1\} = \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ . Since  $P_1 = P_2$  and  $P_1 \in \langle \mathcal{S}_1 \rangle$  and  $P_2 \in \langle \mathcal{Z}'_1 \rangle$ , we have  $P_1 \in \langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_1 \rangle$ . Take any  $E \subseteq \mathcal{Z}_1$  such that  $P_1 \in \langle E \rangle$ . Since  $P \in \langle \{P_1\} \cup \mathcal{S}_2 \rangle \subseteq \langle E \cup \mathcal{S}_2 \rangle$  and  $P \notin \langle \mathcal{U} \rangle$  for any  $\mathcal{U} \subsetneq \mathcal{Z}$ , we get  $E \cup \mathcal{S}_2 = \mathcal{Z}$ . Hence  $E = \mathcal{Z}_1$ . Therefore  $\mathcal{Z}_1$  computes  $\text{sbr}(P_1)$  with respect to  $C_d$ . Similarly,  $\mathcal{Z}'_1$  computes  $\text{sr}(P_2)$  with respect to the same rational normal curve  $C_d$ . Since  $P_1 = P_2$ , we have  $\mathcal{Z}'_1 = \mathcal{Z}_1$  (as for all curves we get the uniqueness of  $\mathcal{Z}_1$  for  $\text{sbr}(P_1) \leq \lfloor (d+2)/2 \rfloor$ ). Since  $\text{sbr}(P_1) \neq \text{sr}(P_1)$ , a theorem of Sylvester gives  $\text{sbr}(P_1) + \text{sr}(P_1) = d + 2$ , i.e.  $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ .

**Definition 1.** A scheme  $Z \subset \mathbb{P}^m$  is said to be in *linearly general position* if for every linear subspace  $R \subsetneq \mathbb{P}^m$  we have  $\deg(R \cap Z) \leq \dim(R) + 1$ .

Notice that the next theorem is false if either  $d = 2$  or  $m = 1$ .

**Theorem 2.** Fix integers  $m \geq 2$  and  $d \geq 4$ . Fix  $P \in \mathbb{P}^N$ . Let  $Z \subset \mathbb{P}^m$  be any smoothable zero-dimensional scheme such that  $P \in \langle \nu_d(Z) \rangle$  and  $P \notin \langle \nu_d(Z') \rangle$ . Assume  $\deg(Z) \leq d$  and that  $Z$  is in linearly general position. Then  $Z$  is the unique scheme  $Z' \subset \mathbb{P}^m$  such that  $\deg(Z') \leq d$  and  $P \in \langle \nu_d(Z') \rangle$ . Moreover  $\nu_d(Z)$  computes  $\text{sbr}(P)$ .

*Proof.* The existence of a scheme computing  $\text{sbr}(P)$  follows from [2], Remark ++= and the assumption “ $\text{sbr}(P) \leq d$ ”. Fix any scheme  $Z' \subset \mathbb{P}^m$  such that  $Z' \neq Z$ ,  $\deg(Z') \leq d$ ,  $P \in \langle \nu_d(Z') \rangle$ , and  $P \notin \langle \nu_d(Z'') \rangle$  for any  $Z'' \subsetneq Z'$ . Assume  $Z' \neq Z$ . Since  $\deg(Z \cup Z') \leq 2d + 1$  and  $h^1(\mathbb{P}^m, \mathcal{I}_{Z \cup Z'}(d)) > 0$  ([2], Lemma 1), there is a line  $D \subset \mathbb{P}^m$  such that  $\deg(D \cap (Z \cup Z')) \geq d + 2$ . Since  $Z$  is in linearly general position and  $m \geq 2$ , we have  $\deg(Z \cap D) \leq 2$ . Hence  $\deg(Z' \cap D) \geq d$ .

Hence  $\deg(Z') = d$ . Since  $\deg(Z') = d$ , we get  $Z' \subset D$ . Hence  $P \in \langle \nu_d(D) \rangle$ . Hence  $\text{sr}(P) = d$ . As for all curves we get  $\text{sbr}(P) \leq \lfloor (d+2)/2 \rfloor$ . Since  $\deg(Z') = d$ , we assumed  $\deg(Z') \leq \text{sbr}(P)$ , contradicting the assumption  $d \geq 4$ .  $\square$

**Corollary 4.** *Fix integers  $m \geq 2$  and  $d \geq 4$ . Fix  $P \in \mathbb{P}^N$  such that  $\text{sbr}(P) \leq d$ . Let  $Z \subset \mathbb{P}^m$  be any smoothable zero-dimensional scheme such that  $\nu_d(Z)$  computes  $\text{sbr}(P)$ . Assume that  $Z$  is in linearly general position. Then  $Z$  is the unique scheme computing  $\text{sbr}(P)$ .*

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